

EPSILON FACTOR FOR $\mathrm{GL}_l \times \mathrm{GL}_{l'}$; $l \neq l'$ PRIMES

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ABSTRACT. Let F be a non-Archimedean local field with finite residual field of characteristic p . In this article we calculate the ε -factor of pairs for $\mathrm{GL}_l(F) \times \mathrm{GL}_{l'}(F)$ where l and l' are distinct primes including the case $l = p$. For this calculation, we use the local Langlands correspondence and non-Galois base change lift. This method leads to the explicit conjecture of the ε -factor of the representations of $\mathrm{GL}_m \times \mathrm{GL}_n$ when n is relatively prime to m and p .

1. INTRODUCTION

Let F be a non-Archimedean local field with finite residual field of characteristic p and the \mathcal{W}_F the absolute Weil group of F . For an integer $n \geq 1$, we denote by $\mathcal{A}_n(F)$ the set of equivalent classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ and by $\mathcal{G}_n(F)$ the set of equivalent classes of irreducible continuous complex representations of \mathcal{W}_F of dimension n . The local Langlands conjecture tells us that there exists a unique bijection Λ_n^F from $\mathcal{G}_n(F)$ to $\mathcal{A}_n(F)$ which satisfies the following conditions:

(1) For $\chi \in \widehat{F^\times}$ and $\sigma \in \mathcal{G}_n(F)$,

$$(1.1) \quad \Lambda_n^F(\chi\sigma) = \chi\Lambda_n^F(\sigma)$$

(By the reciprocity map of local class field theory, we identify $\widehat{F^\times} = \mathcal{A}_1(F)$ with $W_F^{ab} = \mathcal{G}_1(F)$. By this identification, Λ_1 is the identity map.)

(2) For $\sigma \in \mathcal{G}_n(F)$,

$$(1.2) \quad \Lambda_n^F(\check{\sigma}) = \Lambda_n^F(\sigma)^\vee.$$

(3) Let ω_π denote the central quasi-character of $\pi \in \mathcal{A}_n(F)$. For $\sigma \in \mathcal{G}_n(F)$.

$$(1.3) \quad \omega_{\Lambda_n^F(\sigma)} = \det \sigma.$$

(4) Let ψ_F be a non-trivial character of F . For $\sigma \in \mathcal{G}_n(F)$,

$$(1.4) \quad \varepsilon(\Lambda_n^F(\sigma), s, \psi_F) = \varepsilon(\sigma, s, \psi_F).$$

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where the left hand side is the Godement-Jacquet local constant [13] and the right hand side is the Langlands-Deligne local constant [11]. (In fact, this condition is contained in the following condition (5).)

(5) For $\sigma \in \mathcal{G}_n(F)$ and $\sigma' \in \mathcal{G}_{n'}(F)$,

$$(1.5) \quad \varepsilon(\Lambda_n^F(\sigma) \times \Lambda_{n'}^F(\sigma'), s, \psi_F) = \varepsilon(\sigma \otimes \sigma', s, \psi_F)$$

where the ε -factor of pairs of the left hand side is in the sense of [19], [25].

This conjecture has been proved in [23] when $\text{ch } F = p$ and in [14], [17] when $\text{ch } F = 0$. But their proof relies on the global tool and say nothing explicit about the local Langlands correspondence.

On the other hand, there are some explicit correspondences in the following cases:

- (1) When $(n, p) = 1$, Howe-Moy [15], [22] gives an explicit bijection between $\mathcal{G}_n(F)$ and $\mathcal{A}_n(F)$ when $(n, p) = 1$. (See also [24]).
- (2) When $n = p$, Kutzko-Moy [20] gives an explicit bijection between $\mathcal{G}_n(F)$ and $\mathcal{A}_n(F)$. (See also [16]).
- (3) When $n = p^m$, Bushnell-Henniart [3] gives an explicit bijection between $\mathcal{G}_{p^m}^{wr}(F)$ and $\mathcal{A}_{p^m}^{wr}(F)$. (For the definition of $\mathcal{G}_{p^m}^{wr}(F)$ and $\mathcal{A}_{p^m}^{wr}(F)$, see below Remark 3.2.)

All three bijections satisfy the condition (1)–(4) of the local Langlands correspondence. Thus the main obstacle to get the explicit local Langlands correspondence is ε -factor of pairs. We know very few about the explicit calculation of $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$ for $\pi_1 \in \mathcal{A}_{n_1}(F)$ and $\pi_2 \in \mathcal{A}_{n_2}(F)$; The known cases are (i) $n_1 = n_2$ ([21]) and $\pi_2 = \check{\pi}_1$ ([5]), (ii) $\pi_1 \in \mathcal{A}_{p^{i_1}}^{wr}(F)$ and $\pi_2 \in \mathcal{A}_{p^{i_2}}^{wr}(F)$ ([6]).

In this paper we consider the case $n_1 \neq n_2$ are primes. Set $n_1 = l$ and $n_2 = l'$. We admit the case $l = p$. Since $l \neq l'$, we may assume $l' \neq p$. We get the relation of ε -factor of $\text{GL}_l(F) \times \text{GL}_{l'}(F)$ with ε -factor of $\text{GL}_l(E)$ where E is an extension of F associated with π_2 . (See Theorem 4.1.)

Let us summarize the contents of this paper, indicating its organization:

Section 1 reviews the construction of irreducible supercuspidal representations π of $\text{GL}_l(F)$ and the explicit formula of $\varepsilon(\pi, s, \psi_F)$. All of this section is well-known. Section 2 is devoted to review some explicit correspondences and the tame lifting. When $l \neq p$, $\mathcal{G}_l(F)$ consists of the representations in the form $\text{Ind}_{W_E}^{W_F} \theta$; E/F is an extension of degree l and θ is a quasi-character of E^\times . By way of such θ , there is very explicit Howe-Moy correspondence between $\mathcal{G}_l(F)$ and $\mathcal{A}_l(F)$. But when $l = p$, there exists non-monomial representations in $\mathcal{G}_p(F)$; so we need the tame base change lift to get the correspondence. (See [20], [3].) Let $\pi \in \mathcal{A}_p^{wr}(F)$ and K/F a tamely ramified extension. After the definition

of [2], we give the tame base change lift $l_K(\pi)$ explicitly (Theorem 3.5) and show l_K is compatible with the local Langlands correspondence (Proposition 3.7). We also define the tame base change lift l_K for the case $l \neq p$ and prove it is compatible with the Howe-Moy correspondence (Proposition 3.9). These are essential tool to calculate the ϵ -factor of pairs. Section 3 calculates the ϵ -factor of $\mathrm{GL}_{l'}(F) \times \mathrm{GL}_l(F)$. By the result of Bushnell-Henniart [7], the Howe-Moy correspondence coincides with the Local Langlands correspondence for $\mathrm{GL}_l(F)$. Thus we calculate the ϵ -factor in the Galois side and then transfer it to the automorphic side using the results in section 2.

Notation

Let F be a non-archimedean local field. We denote by \mathcal{O}_F , P_F , ϖ_F , k_F and v_F the maximal order of F , the maximal ideal of \mathcal{O}_F , a prime element of P_F , the residue field of F and the valuation of F normalized by $v_F(\varpi_F) = 1$. We set $q = q_F$ to be the number of elements in k_F . Let W_F be the absolute Weil group of F . Hereafter we fix an additive character ψ of F whose conductor is P_F , i.e., ψ is trivial on P_F and not trivial on \mathcal{O}_F . For an extension E over F , we denote by tr_E , N_E the trace and norm to F respectively. We set $\psi_E = \psi \circ \mathrm{tr}_E$ and $\chi_E = \chi \circ N_E$ for a quasi-character χ of F^\times . Let θ be a quasi-character of E^\times . We denote by $f(\theta)$ an integer such that $1 + P_E^{n+1} \not\subset \mathrm{Ker} \theta$ and $1 + P_E^n \subset \mathrm{Ker} \theta$. The Gauss sum $G(\theta, \psi_E)$ is defined by

$$(1.6) \quad G(\theta, \psi_E) = \begin{cases} q_E^{-1/2} \sum_{x \in k_E^\times} \theta^{-1}(x) \psi_E(x) & \text{if } f(\theta) = 1 \\ q_E^{-1/2} \sum_{x \in k_E} \theta^{-1}(1 + \varpi_E^m x) \psi_E(\varpi_E^m x) & \text{if } f(\theta) = 2m + 1. \end{cases}$$

The λ -factor λ_E of E/F is defined by

$$(1.7) \quad \lambda_E = \frac{\varepsilon(\mathrm{Ind}_{W_E}^{W_F} 1_{W_K}, s, \psi_F)}{\varepsilon(1_{W_K}, \psi_E)}.$$

It is well-known that

$$(1.8) \quad \varepsilon(\mathrm{Ind}_{W_E}^{W_F} \sigma, s, \psi_F) = \lambda_E^{\dim \sigma} \varepsilon(\sigma, s, \psi_E)$$

for any representation σ of W_E . The trace of matrix is denoted by Tr . For an irreducible admissible representation π of $\mathrm{GL}_l(F)$, the conductor exponent of π is defined to be the integer $f(\pi)$ such that the local constant $\varepsilon(s, \pi, \psi)$ of Godement-Jacquet [13] is the form $aq^{-s(f(\pi)-l)}$.

Let G be a totally disconnected, locally compact group. We denote by \widehat{G} the set of (equivalence classes of) irreducible admissible representations of G . For a closed subgroup H of G and a representation ρ of H , we denote by $\mathrm{Ind}_H^G \rho$ (resp. $\mathrm{ind}_H^G \rho$) the induced representation (resp. compactly induced representation) of ρ to G . For a representation π of G , we denote by $\pi|_H$ the restriction of π to H .

2. CONSTRUCTION OF THE REPRESENTATION $\mathrm{GL}_l(F)$

Let l be an arbitrary prime number (we allow the case $l = p$). We set $V_F = F^l$ so that $M_l(F) = \mathrm{End}_F(V_F)$ and $\mathrm{GL}_l(F) = \mathrm{Aut}_F(V_F)$. Throughout this paper, we write $G = G_F = \mathrm{GL}_l(F)$ and $G_K = \mathrm{GL}_l(K)$. In this section, we review the construction of the supercuspidal representation of $\mathrm{GL}_l(F)$ and its lift to $\mathrm{GL}_l(K)$ where K/F is a tamely ramified extension. Most of the contents of this section are well-known (See [8],[22] and [3].)

Definition 2.1. Let $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ be the set of \mathcal{O}_F -lattices in V_F . \mathcal{L} is said to be a uniform lattice chain of period $e = e(\mathcal{L})$ if the following conditions hold for all $i \in \mathbb{Z}$:

- (1) $L_{i+1} \subset L_i$,
- (2) $P_F L_i = L_{i+e}$,
- (3) $\dim_{k_F}(L_i/L_{i+1}) = l/e$.

Since we assume l is a prime, the period $e(\mathcal{L})$ is either l or 1.

Definition 2.2. For a uniform lattice chain $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ of period e , we set

$$\mathfrak{A}(\mathcal{L}) = \{f \in M_l(F) \mid f(L_i) \subset L_i \text{ for all } i\},$$

Then $\mathfrak{A}(\mathcal{L})$ is a principal order in $M_l(F)$ and its Jacobson radical $\mathfrak{P}(\mathcal{L})$ is

$$\{f \in M_l(F) \mid f(L_i) \subset L_{i+1} \text{ for all } i\}.$$

We also set the period $e(\mathfrak{A})$ of \mathfrak{A} is the period of \mathcal{L} . Put $U(\mathfrak{A}) = \mathfrak{A}^\times$, $U(\mathfrak{A})^n = 1 + \mathfrak{P}^n$ for any positive integer n and

$$\mathfrak{K}(\mathfrak{A}) = \mathrm{Aut}(\mathcal{L}) = \{x \in \mathrm{GL}_l(F) \mid x^{-1}\mathfrak{A}x = \mathfrak{A}\}.$$

By taking an appropriate \mathcal{O}_F -basis of L_0 , we express the principal orders by the following matrix form. If $e(\mathcal{L}) = l$, \mathfrak{A} (resp. $\mathfrak{P}(\mathcal{L})$) is G -conjugate to $M_l(\mathcal{O}_F)$ (resp. $M_l(P_F)$). When $e(\mathcal{L}) = 1$, up to G -conjugacy,

$$\mathfrak{A}(\mathcal{L}) = \left\{ \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \\ P_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \\ \cdots & \cdots & \cdots & \cdots \\ P_F & P_F & \cdots & \mathcal{O}_F \end{pmatrix} \right\}$$

and

$$\mathfrak{P}(\mathcal{L}) = \left\{ \begin{pmatrix} P_F & \mathcal{O}_F & \cdots & \mathcal{O}_F \\ P_F & P_F & \cdots & \mathcal{O}_F \\ \cdots & \cdots & \cdots & \cdots \\ P_F & P_F & \cdots & P_F \end{pmatrix} \right\}.$$

Let r, n be integers satisfying

$$n > r \geq \left\lceil \frac{n}{2} \right\rceil \geq 0,$$

where $[x]$ denote the greatest integer $\leq x$. For $\beta \in \mathrm{M}_l(F)$, we define a function ψ_β on $U(\mathfrak{A})^r$ by

$$(2.1) \quad \psi_\beta(1+x) = \psi(\mathrm{Tr} \beta x).$$

Then the map $u \mapsto \psi_\beta$ induces an isomorphism between $\mathfrak{P}^{-r+1}/\mathfrak{P}^{-n+1}$ and the complex dual, $(U(\mathfrak{A})^r/U(\mathfrak{A})^n)^\wedge$, of $U(\mathfrak{A})^r/U(\mathfrak{A})^n$.

Definition 2.3. Let E/F be a field extension in $\mathrm{M}_l(F)$. An element $\beta \in E$ is said to be E/F -minimal if the following conditions hold:

- (1) $(v_E(\beta), e(E/F)) = 1$.
- (2) $k_F(\varpi_F^{-v_E(\beta)} \beta^{e(E/F)} \bmod P_E) = k_E$.

When $E \subset \mathrm{M}_L(F)$ and $E \neq F$, E/F is an extension of degree l since l is a prime. Thus we can identify E with V_F . By this identification, $\{P_E^i\}_{i \in \mathbb{Z}}$ becomes a uniform lattice chain of period $e(E/F)$. We put $\mathfrak{A}(E) = \mathfrak{A}(P_E^i)$.

Proposition 2.4. Suppose β is E/F -minimal and $E \neq F$. For $\mathfrak{A} = \mathfrak{A}(E)$, we have:

- (1) $\mathfrak{K}(\mathfrak{A}) = E^\times U(\mathfrak{A})$ and $E^\times \cap U(\mathfrak{A}) = \mathcal{O}_E^\times$.
- (2) $E \cap \mathfrak{P}^m = P_E^m$ for all integers m and $E^\times \cap U(\mathfrak{A})^m = 1 + P_E^m$ for all integers $m \geq 1$.
- (3) Let $x \in \mathfrak{P}^l$. If $\beta x - x\beta \in \mathfrak{P}^{m+l+1}$, then $x \in E + \mathfrak{P}^{l+1}$.

Proof. The last assertion of the above proposition is due to Carayol (see [8]). The rest is obvious. \square

We shall construct the irreducible supercuspidal representations of $\mathrm{GL}_l(F)$ from E/F -minimal elements. Let E/F be a field extension of degree l , β an E/F -minimal element and $\mathfrak{A} = \mathfrak{A}(E)$. Put $v_E(\beta) = 1 - n < 0$ and $m = [n/2]$. Then ψ_β is a quasi-character of $U(\mathfrak{A})^m$ whose kernel contains $U(\mathfrak{A})^n$. Put $H = E^\times U(\mathfrak{A})^m$ and define a quasi-character $\rho_{\beta, \theta}$ of H by

$$(2.2) \quad \rho_{\beta, \theta}(h \cdot g) = \theta(h) \psi_\beta(g) \quad \text{for } h \in E^\times, \quad g \in U(\mathfrak{A})^m$$

where θ is a quasi-character of E^\times such that $\theta|_{E^\times \cap U(\mathfrak{A})^m} = \psi_\beta|_{E^\times \cap U(\mathfrak{A})^m}$. We note $f(\theta) = 1 - v_E(\beta) = n$ where $f(\theta)$ is the exponent of the conductor of θ i.e. the minimum integer such that $\mathrm{Ker} \theta \subset 1 + P_E^n$.

Put J be the normalizer of ψ_β in $\mathfrak{K}(\mathfrak{A})$ i.e.

$$J = \{a \in \mathfrak{K}(\mathfrak{A}) \mid \psi_\beta^a = \psi_\beta\}$$

where $\psi_\beta^a(x) = \psi_\beta(a^{-1}xa)$ for $x \in U(\mathfrak{A})^m$. Then $J = E^\times U(\mathfrak{A})^{m'}$ where $m' = [n/2]$ by virtue of Proposition 2.4. Put $\eta_{\beta, \theta} = \mathrm{Ind}_H^{\mathfrak{K}(\mathfrak{A})} \rho_{\beta, \theta}$.

When n is even, i.e. $n = 2m$, then $J = H = E^\times U(\mathfrak{A})^m$. By the Clifford theory, $\eta_{\beta, \theta}$ is an irreducible representation of $\mathfrak{K}(\mathfrak{A})$. We put

$$(2.3) \quad \kappa_{\beta, \theta} = \eta_{\beta, \theta}.$$

When n is odd, i.e. $n = 2m - 1$, then $J = E^\times U(\mathfrak{A})^{m-1}$. Thus $\eta_{\beta,\theta}$ is not an irreducible representation of $\mathfrak{K}(\mathfrak{A})$. Even in this case, we can describe the irreducible component of $\eta_{\beta,\theta}$ by β and θ . If E/F is unramified, we put

$$(2.4) \quad \kappa_{\beta,\theta} = \frac{(-1)^l (q^{l(l-1)/2} - (-1)^{l-1})(q-1)}{q^{l(l-1)/2}(q^l-1)} \sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \eta_{\beta,\theta \otimes \chi} + (-1)^{l-1} \eta_{\beta,\theta}.$$

Now we assume we treat the case E/F is ramified. If $l \neq p$, we put

$$(2.5) \quad \kappa_{\beta,\theta} = \frac{1 - \left(\frac{q}{l}\right) q^{(l-1)/2}}{l q^{(l-1)/2}} \sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \eta_{\beta,\theta \otimes \chi} + \left(\frac{q}{l}\right) \eta_{\beta,\theta}$$

where $\left(\frac{q}{l}\right)$ is the Legendre symbol. By Lemma 3.5.30 and Lemma 3.5.33 in [22], the virtual representation $\kappa_{\beta,\theta}$ is an irreducible component of η_θ .

Next we treat the case $l = p$. If f is odd, we put

$$(2.6) \quad \kappa_\theta = \sum_{i=0}^{p-1} \left(\frac{1}{p q^{(l-1)/2}} + \left(\frac{i}{p}\right) \right) \frac{p^{(f-1)/2}}{G_0 G(\beta)} \eta_{\theta \otimes \chi^i}$$

where χ is a generator of $(E^\times/F^\times(1+P_E))^\wedge$ determined by $\chi(\varpi_E) = \exp(2\pi\sqrt{-1}/p)$ and $G_0, G(\beta)$ are Gauss sums defined by

$$(2.7) \quad G(\beta) = \frac{1}{\sqrt{q}} \sum_{x \in k_E} \psi(\text{tr}_{k_E/k_F} \frac{1}{2} \beta \varpi_E^{2(m-1)} (-1)^{(p+1)/2} x^2)$$

$$(2.8) \quad G_0 = \frac{1}{\sqrt{p}} \sum_{a=1}^l \left(\frac{a}{p}\right) \exp(2\pi\sqrt{-1}a/p).$$

When f is even, we put

$$(2.9) \quad \kappa_\theta = \sum_{\chi \in (E^\times/F^\times(1+P_E))^\wedge} \frac{q^{1/2} G(\beta) - q^{(p-1)/2}}{G(\beta) p q^{p/2}} \eta_{\theta \otimes \chi} + \frac{1}{q^{1/2} G(\beta)} \eta_\chi.$$

By Proposition 3.4 in [27], κ_θ is an irreducible component of η_θ .

Finally we consider the level 1 supercuspidal representation. Let E/F be an unramified extension of degree l , θ a quasi-character of E^\times which is trivial on $1 + P_E$ and $\mathfrak{A} = \mathfrak{A}(E)$. Then there is an irreducible representation κ_θ of $U(\mathfrak{A})$ which is trivial on $U(\mathfrak{A})^1$ and its tensor product with the pull-back of the Steinberg representation of $U(\mathfrak{A})/U(\mathfrak{A})^1 \simeq \text{GL}_l(k_F)$ is the representation induced by the one-dimensional representation $tx \mapsto \theta(t), t \in \mathcal{O}_E^\times, x \in U(\mathfrak{A})^1$, of the subgroup $\mathcal{O}_E^\times U(\mathfrak{A})^1$. We denote by κ_θ the representation $tx \mapsto \theta(t) \kappa_\theta(x), t \in F^\times, x \in U(\mathfrak{A})$, of $\mathfrak{K}(\mathfrak{A})$.

Theorem 2.5. *Let the notation be as above. Then $\kappa_{\beta, \theta}$ and κ_θ are irreducible representations of $\mathfrak{K}(\mathfrak{A})$. Put $\pi_F(\beta, \theta) = \mathrm{ind}_{\mathfrak{K}(\mathfrak{A})}^G \kappa_{\beta, \theta}$ and $\pi_F(\theta) = \mathrm{ind}_{\mathfrak{K}(\mathfrak{A})}^G \kappa_\theta$. Then $\pi_F(\beta, \theta)$ and $\pi_F(\theta)$ are irreducible supercuspidal representations of $G = \mathrm{GL}_l(F)$ with $f(\pi_F(u, \theta)) = f(E/F)(f(\theta) - 1) + l$ and $f(\pi_F(\theta)) = lf(\theta)$. Every irreducible supercuspidal representation of G can be written in the form $\chi\pi_F(\beta, \theta)$ or $\chi\pi_F(\theta)$ for some E/F -minimal element $\beta, \theta \in \widehat{F(\beta)^\times}$ and $\chi \in \widehat{F^\times}$.*

The ε -factors of all supercuspidal representations of G have been calculated completely. (See [22],[20]).

Theorem 2.6. *Let $\pi_F(\beta, \theta)$ and $\pi_F(\theta)$ be as above. Put $n = f(\theta)$. For $\chi \in \widehat{F^\times}$, we pick an element $c_\chi \in F$ such that $\chi(1+x) = \psi_F(c_\chi x)$ for $x \in P_F^{[(f(\chi)+1)/2]}$. (If $f(\chi) \leq 1$, we take $c_\chi = 0$). Put $n(\chi) = \max(n, e(E/F)(f(\chi) - 1) + 1)$ and $\beta_\chi = \beta + c_\chi$.*

(1) *If $n(\chi)$ is even,*

$$\varepsilon(\chi\pi_F(\beta, \theta), s, \psi) = \psi_E(\beta_\chi)(\chi_E\theta)(\beta_\chi^{-1})|\beta_\chi|_E^s.$$

(2) *If $n(\chi) = n = 1$,*

$$\varepsilon(\chi\pi_F(\theta), s, \psi) = (-1)^{l-1}\varepsilon(\chi_E\theta, s, \psi_E).$$

(3) *If $n(\chi) \neq 1$ is odd,*

$$\varepsilon(\chi\pi_F(\beta, \theta), s, \psi) = \psi_E(\beta_\chi)(\chi_E\theta)(\beta_\chi^{-1})|\beta_\chi|_E^s G$$

where the Gauss sum G is defined by

$$G = \begin{cases} G(\theta, \psi_E) & \text{if } n = n(\chi) \text{ and } E/F \text{ is tamely ramified} \\ G(\beta) & \text{if } n = n(\chi) \text{ and } E/F \text{ is wildly ramified} \\ \lambda_E G(\chi, \psi_F)^l & \text{if } n > n(\chi) \end{cases}$$

where λ_E is defined in (1.7).

3. EXPLICIT CORRESPONDENCES AND TAME BASE CHANGE LIFT

Now we consider some correspondences between $\mathcal{A}_F(l)$ and $\mathcal{G}_F(l)$ which satisfy the conditions (i)-(iv) of the local Langlands correspondence. When $l \neq p$ or $l = p$ and E/F is unramified, this is a special case of Howe-Moy correspondence.

Definition 3.1. A quasi-character θ of E^\times is called generic if $f(\theta) \not\equiv 1 \pmod{l}$. For a generic character θ of E^\times , $\beta_\theta \in P_E^{1-f(\theta)} - P_E^{2-f(\theta)}$ is defined by

$$(3.1) \quad \theta(1+x) = \psi_E(\beta_\theta x) \quad \text{for } x \in P_E^{[(f(\theta)+1)/2]}.$$

Then β is E/F -minimal. We denote by $\widehat{E_{gen}^\times}$ the set of generic quasi-characters of E^\times .

Remark 3.2. When E/F is tamely ramified, the generic quasi-character θ determines uniquely $\pi_F(\beta_\theta, \theta)$. (See [22]). In this case we simply denote $\pi_F(\beta, \theta)$ by $\pi_F(\theta)$. When $l = p$, we need β to determine the representation $\pi_F(\beta, \theta)$.

To separate the wildly ramified case, we introduce some notations. Let $\mathcal{A}_l^{wr}(F)$ denote the set $\pi = \chi\pi_F(\beta, \theta) \in \mathcal{A}_l(F)$ with the property that $F(\beta)/F$ is wildly ramified. $\pi \in \mathcal{A}_l^{wr}(F)$ is equivalent to $l = p$ and $\pi \simeq \chi\pi$ for some unramified quasi-character $\chi \neq 1$ of F^\times . We put $\mathcal{A}_l^t(F) = \mathcal{A}_l(F) \setminus \mathcal{A}_l^{wr}(F)$. Similarly let $\mathcal{G}_l^{wr}(F)$ denote the set $\sigma \in \mathcal{G}_l(F)$ with the property that $\sigma \otimes \chi$ is equivalent to π for some unramified quasi-character $\chi \neq 1$ of F^\times and $l = p$. We also put $\mathcal{G}_l^t(F) = \mathcal{G}_l(F) \setminus \mathcal{G}_l^{wr}(F)$. If $p \neq l$, $\mathcal{A}_l(F) = \mathcal{A}_l^t(F)$ and $\mathcal{G}_l(F) = \mathcal{G}_l^t(F)$. The Howe-Moy correspondence gives a bijection between $\mathcal{G}_l^t(F) = \mathcal{A}_l^t(F)$. (See [22] and [12].)

If E/F is tamely ramified, λ_E is easily calculated.

Lemma 3.3. *Let E/F is a tamely ramified extension of degree l . Then*

$$\lambda_E = \begin{cases} (-1)^{l-1} & \text{if } e(E/F) = 1, \\ \left(\frac{q}{l}\right) & \text{if } e(E/F) = l \neq 2 \\ q^{-1/2} \sum_{x \in k_E} \text{sgn}_{E/F}^{-1}(x) \psi_E(x) & \text{if } e(E/F) = l = 2 \end{cases}$$

Proof. See (2.5.3), (2.5.5) and Proposition 2.5.11 of [22]. \square

Theorem 3.4. *Let E be a tamely ramified extension of F of degree l and θ be a generic quasi-character of E^\times . We define a quasi-character δ_E of E^\times as follows:*

When $e(E/F) \neq 2$, $\delta_E(x) = \lambda_E^{v_E(x)}$.

When $e(E/F) = 2$,

$$\delta_E(x) = \begin{cases} 1 & \text{if } x \in 1 + P_E, \\ \text{sgn}_{E/F}(x) & \text{if } x \in F^\times, \\ \lambda_E & \text{if } x = \beta_\theta. \end{cases}$$

We set

$$\sigma_F(\theta) = \delta_E \text{Ind}_{W_E}^{W_F}.$$

- (1) *the representation $\sigma_F(\theta)$ belongs to $\mathcal{G}_l^t(F)$ and any element of $\mathcal{G}_l^t(F)$ can be written in the form $\chi\sigma_F(\theta)$ for an extension E/F of degree l , a generic character of E^\times and a quasi-character χ of F^\times .*
- (2) *Define the map Φ_l^F from $\mathcal{G}_l^t(F)$ to $\mathcal{A}_l^t(F)$ by*

$$\Phi_l^F(\chi\sigma_F(\theta)) = \chi\pi_F(\delta_E\theta).$$

Then Φ_l^F is a bijection which satisfies the following conditions:

- (a) *For $\chi \in \widehat{F^\times}$ and $\sigma \in \mathcal{G}_l^t(F)$,*

$$\Phi_l^F(\chi\sigma) = \chi\Phi_l^F(\sigma).$$

(b) For $\sigma \in \mathcal{G}_l^t(F)$,

$$\Phi_l^F(\check{\sigma}) = \Phi_l^F(\sigma)^\vee.$$

(c) Let ω_π denote the central quasi-character of $\pi \in \mathcal{A}_l(F)$.
For $\sigma \in \mathcal{G}_l^t(F)$.

$$\omega_{\Phi_l^F(\sigma)} = \det \sigma.$$

(d) For $\sigma \in \mathcal{G}_l^t(F)$,

$$\varepsilon(\Phi_l^F(\sigma), s, \psi_F) = \varepsilon(\sigma, s, \psi_F).$$

Since $\mathcal{G}_p^{wr}(F)$ contains non-monomial representations, the correspondence between $\mathcal{G}_p^{wr}(F)$ and $\mathcal{A}_p^{wr}(F)$ becomes more complicated. We use the tame lifting of Bushnell-Henniart [3]. For any tamely ramified extension K/F , including the case K/F is non-Galois, the tame lifting map \mathbf{l}_K from $\mathcal{A}_{p^i}^{wr}(F)$ to $\mathcal{A}_{p^i}^{wr}(K)$ is constructed by Bushnell-Henniart. Since we consider the case $i = 1$, this base change is easy to describe. Since K/F is tamely ramified, $E \otimes_F K = EK$ is an extension of field of K , $G_K = G(K)$ can be identified with $\mathrm{Aut}_K(E \otimes_F K)$ and $\beta = \beta \otimes 1$ becomes an EK/K -minimal element in $V_K = \mathrm{End}_K(EK)$. Moreover if θ is a quasi-character of E^\times such that $\theta(1+x) = \psi(\mathrm{tr}_{E/F} \beta x)$ for $x \in P_E^m$, then $\theta \circ n_{EK/E}(1+x) = \psi_K(\mathrm{tr}_{EK/K} \beta x)$ for $x \in P_{EK}^m$. Therefore we get an irreducible supercuspidal representation $\pi_K(\beta, \theta \circ N_{EK/E}) \in \mathcal{A}_p^{wr}(K)$.

Theorem 3.5. *Let K/F be an extension of degree prime to p and \mathbf{l}_K the lifting from $\mathcal{A}_p^{wr}(F)$ to $\mathcal{A}_p^{wr}(K)$ defined by (5.3.3) in [3]. Put $\Delta_K = \det \mathrm{Ind}_{W_K}^{W_F} 1_{W_K} \in \widehat{F^\times}$ and $\tilde{\Delta} = \Delta_K \circ N_{E/F} \in \widehat{E^\times}$. For $\pi_F(\beta, \theta) \in \mathcal{A}_p^{wr}(F)$ and $\chi \in \widehat{F^\times}$, we have:*

$$\begin{aligned} \mathbf{l}_K(\chi \pi_F(\beta, \theta)) &= \chi_K \pi_K(\beta, (\tilde{\Delta}^{e(E/F)-1} \theta) \circ N_{EK/E}) \\ &= \begin{cases} \chi_K \pi_K(\beta, \theta \circ N_{EK/E}) & e(E/F) \neq 2 \\ \chi_K \pi_K(\beta, (\tilde{\Delta} \theta) \circ N_{EK/E}) & e(E/F) = 2. \end{cases} \end{aligned}$$

Proof. Since two lifting maps are compatible with twist of quasi-character of F^\times , we may assume $\chi = 1$. By Proposition 10.2 of [3], it suffices to say

$$\varepsilon(\mathbf{l}_K(\pi_F(\beta, \theta)), s, \psi_K) = \varepsilon(\pi_K(\beta, (\tilde{\Delta}^{e(E/F)-1} \theta) \circ N_{EK/E}), s, \psi_K).$$

(Other conditions (a) and (b) in Proposition 10.2 of [3] are obvious in our case.) Theorem 1.6 of [3] tells us that

$$\lambda_K^p \varepsilon(\mathbf{l}_K(\pi_F(\beta, \theta)), s, \psi_K) = \Delta(N_{E/F}(\beta)) \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}.$$

On the other hand, it follows from Proposition 2.2.11 of [20] that

$$\lambda_K^p \varepsilon(\pi_K(\beta, \theta \circ N_{EK/E}), s, \psi_K) = \Delta(N_{E/F}(\beta)) \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}$$

if $p \neq 2$. (Proposition 2.2.11 of [20] assumes K/F is Galois, but it holds including the case K/F is non-Galois since Proposition 2.5.16 of [22] holds for any tamely ramified extension K/F .) Hence the assertion holds when $p \neq 2$. When $p = 2$, $n(\pi_F(\beta, \theta) = 1 - v_E(\beta)$ is even since $(v_E(\beta), p) = 1$. Therefore Theorem 2.6 tells us

$$\varepsilon(\pi_K(\beta, \theta \circ N_{EK/E}), s, \psi_K) = \varepsilon(\pi_F(\beta, \theta), s, \psi_F)^{[K:F]}.$$

Since $\Delta_K \circ N_{K/F}$ is unramified and $\Delta_K^{-1} = \Delta_K$,

$$\varepsilon(\pi_K(\beta, (\tilde{\Delta}\theta) \circ N_{EK/E}), s, \psi_K) = \Delta_K \circ N_{K/F}(\beta) \varepsilon(\pi_K(\beta, \theta \circ N_{EK/E}), s, \psi_K).$$

Hence our assertion holds. \square

Remark 3.6. Two quasi-characters Δ_K and δ_K is closely related. If $e(K/F)$ is odd., $\Delta_K \circ N_{K/F} = \delta_K$. (See Corollary 2.5.5 of [22].)

Using the tame lifting map \mathbf{l}_K , Bushnell-Henniart ([3]) has constructed the correspondence $\mathcal{G}_p^{wr}(F)$ to $\mathcal{A}_p^{wr}(F)$. For $i = 1$, this map coincides with the local Langlands correspondence and is compatible with \mathbf{l}_K . This follows as a special case of Lemma 5.2 in [4].

Proposition 3.7. *Let Λ_l^F be the local Langlands map. Then for any tamely ramified extension K/F and $\sigma \in \mathcal{G}_p^{wr}(F)$, we have:*

$$\mathbf{l}_K \Lambda_l^F(\sigma) = \Lambda_l^K(\sigma|_{W_K}).$$

Proof. By Lemma 5.2 in [4], it suffices to say that the exponent $f(\pi_{\beta, \theta})$ of the conductor of $\pi_{\beta, \theta} \in \mathcal{A}_p^{wr}(F)$ is prime to p . It follows from the fact that $f(\pi_{\beta, \theta}) \equiv -v_E(\beta) \pmod{p}$. \square

We define the lift \mathbf{l}_K for $\pi \in \mathcal{A}_l^t(F)$ as in the case $\pi \in \mathcal{A}_l^{wr}(F)$.

Definition 3.8. Let E/F be an extension of degree l , $\theta \in \widehat{E_{gen}^\times}$ and $\chi \in \widehat{F^\times}$. Assume K is a tamely ramified extension of F such that $([K:F], l) = 1$. Then we define $\mathbf{l}_K(\chi \pi_F(\theta))$ by

$$\begin{aligned} \mathbf{l}_K(\chi \pi_F(\beta, \theta)) &= \chi_K(\Delta_K \circ N_{K/F})^{e(E/F)-1} \pi_K(\beta, \theta \circ N_{EK/E}) \\ &= \begin{cases} \chi_K \pi_K(\beta, \theta \circ N_{EK/E}) & e(E/F) \neq 2 \\ \chi_K(\Delta_K \circ N_{K/F}) \pi_K(\beta, \theta \circ N_{EK/E}) & e(E/F) = 2. \end{cases} \end{aligned}$$

This lifting is compatible with Φ_l .

Proposition 3.9. *Let K/F be a finite, tamely ramified extension satisfying $K \cap E = F$. For $\sigma \in \mathcal{G}_l^t(F)$,*

$$\mathbf{l}_K \Phi_l^F(\sigma) = \Phi_l^K(\sigma|_{W_K}).$$

Proof. Since \mathbf{l}_K and Φ_l are compatible with quasi-character twist, we may assume $\sigma = \sigma_F(\theta)$ for $\theta \in \widehat{E_{gen}^\times}$. By the definition of \mathbf{l}_K and Φ_l ,

$$(\Phi_l^K)^{-1}(\mathbf{l}_K(\Lambda_l^F(\sigma_F(\theta)))) = \text{Ind}_{W_{EK}}^{W_K} \delta_{EK/K}((\tilde{\Delta}^{e(E/F)-1} \theta) \circ N_{EK/K}).$$

On the other hand, it follows from $W_E W_K = W_F$ and $W_E \cap W_K = W_{EK}$ that Mackey's Theorem tells us

$$\sigma_F(\theta)|_{W_K} = \text{Ind}_{W_{EK}}^{W_K}(\delta_E \theta) \circ N_{EK/E}.$$

Thus it suffices to say that

$$(3.2) \quad \delta_{EK/K}(\tilde{\Delta}^{e(E/F)-1} \circ N_{EK/K}) = \delta_E \circ N_{EK/E}.$$

When $e(E/F)$ is odd, this is obvious. So we assume $e(E/F) = 2$. By the definition of δ , we have only to show (3.2) for $x \in K^\times$ and β . For $x \in K^\times$,

$$\begin{aligned} \delta_{EK/K}(\tilde{\Delta} \circ N_{EK/K}(x)) &= \delta_{EK/K}(x) \Delta_K \circ N_{K/F}(x^2) \\ &= \text{sgn}_{EK/K}(x). \end{aligned}$$

since Δ_K has at most order 2. The right hand side of (3.2) becomes $\text{sgn}_{E/F}(N_{EK/E}(x))$, which equals to $\text{sgn}_{EK/K}(x)$ since $[K : F]$ is odd. We compare the value of both sides of (3.2) at β . The left hand side is $\delta_{EK/K}(\beta) \Delta_K(N_{E/F}(\beta))^{[K:F]}$. It follows from Remark 3.6 that $\Delta_K(N_{E/F}(\beta)) = \delta_K^{v_K(\beta)}$. Thus it amounts to $\lambda_{EK/K} \lambda_K$. The right hand side becomes $\lambda_E^{[K:F]}$. After all, the equation $\lambda_{EK/E} \lambda_K = 1$ gives the result. When $[K : F]$ is prime, it follows from Lemma 3.3. The composite case is obtained by the transitivity property of λ -factor. \square

We need to show that the Howe-Moy correspondence $\Phi_{l'}$ coincides with the Local Langlands correspondence $\Lambda_{l'}$.

Theorem 3.10. *For any prime $l' \neq p$,*

$$\Phi_{l'}^F = \Lambda_{l'}^F.$$

Proof. If $l' = 2$, it follows from Converse Theorem ([9]). We assume l' is an odd prime. Let $\pi \in \mathcal{A}_F(l')$. Then there exist an extension E/F of degree l' , $\theta \in \widehat{(E^\times)_{\text{gen}}}$ and $\chi \in \widehat{F^\times}$ such that $\pi = \chi_E \pi_F(\theta)$ as in Remark 3.2. When E/F is unramified, Theorem 9.2 ([26]) implies $\Phi_{l'}^F(\pi) = \chi_E \text{Ind}_{W_E}^{W_F} \theta = \Lambda_{l'}^F(\pi)$. When E/F is ramified, the assertion follows from Theorem B in [7]. \square

Remark 3.11. Theorem 9.2 ([26]) is proved under the assumption $p > l$, but this assumption is dispensable. The key point is to prove that

$$\Theta_\pi^\kappa(x) = \Theta_\pi(x) \quad \text{for } x \in E^\times \setminus F^\times(1 + P_E^r)$$

where Θ_π is a distribution character of π and Θ_π^κ is a κ -twisted distribution character of π for $\kappa \in (\widehat{F^\times/n_{E/F}(E^\times)})$. This is proved in Theorem 6.1 ([26]) without using the assumption $p > l$.

By Propostion 3.7, Propostion 3.9 and Theorem 3.10, Φ_l is comatible with **1** for any prime l .

Corollary 3.12. *Let K/F be a finite, tamely ramified extension satisfying $K \cap E = F$. For any prime l and $\sigma \in \mathcal{G}_l(F)$,*

$$\mathbf{1}_K \Phi_l^F(\sigma) = \Phi_l^K(\sigma|_{W_K}).$$

4. ε -FACTOR OF PAIRS

In this section, we consider the ε -factor $\varepsilon(\pi_1 \times \pi_2, s, \psi_F)$. Let l' be a prime not equal to l and p . We treat the case $\pi_1 \in \mathcal{A}_F(l)$ and $\pi_2 \in \mathcal{A}_F(l')$. Since the local Langlands correspondence and the Bushnell-Henniart base change lift are compatible with quasi-character twists, we may assume π_1 and π_2 are minimal.

Theorem 4.1. *Let $\pi_1 \in \mathcal{A}_F(l)$ and $\pi_2 \in \mathcal{A}_F(l')$ where l' is a prime not equal to l and p . Let E_2/F be an extension of degree l' , $\theta_2 \in \widehat{(E_2^\times)}_{\text{gen}}$ and $\chi_2 \in \widehat{F^\times}$ such that $\pi_2 = (\chi_2)_{E_2} \pi_F(\theta_2)$ as in Remark 3.2. Then we have*

$$(4.1) \quad \varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \lambda_{E_2} \varepsilon(\chi_2 \delta_{E_2} \theta_2 \mathbf{1}_{E_2}(\pi_1), s, \psi_{E_2}).$$

Proof. It follows from $\Phi_{l'}^F = \Lambda_{l'}^F$ that

$$(\Lambda_{l'}^F)^{-1}(\pi_F((\chi_2)_{E_2}(\theta_2))) = \text{Ind}_{W_{E_2}}^{W_F}(\chi_2 \delta_{E_2} \theta_2).$$

Put $(\Lambda_l^F)^{-1}(\pi_1) = \sigma_1$. Then we have:

$$\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \varepsilon(\sigma_1 \otimes \text{Ind}_{W_{E_2}}^{W_F}(\chi_2 \delta_{E_2} \theta_2), s, \psi_F).$$

Since

$$\text{Ind}_{W_{E_2}}^{W_F} \sigma_1 \otimes \chi_2 \delta_{E_2} \theta_2 = \text{Ind}_{W_{E_2}}^{W_F}(\sigma_1|_{W_{E_2}} \otimes \chi_2 \delta_{E_2} \theta_2)$$

and

$$\varepsilon(\text{Ind}_{W_{E_2}}^{W_F} \sigma, s, \psi_F) = \lambda_{E_2}^{\dim \sigma} \varepsilon(\sigma, s, \psi_{E_1}) \quad \text{for } \sigma \in \mathcal{G}_{E_2}(l'),$$

we obtain

$$\begin{aligned} \varepsilon(\pi_1 \times \pi_2, s, \psi_F) &= \varepsilon(\sigma_1|_{W_{E_2}} \otimes \text{Ind}_{W_{E_2}}^{W_F}(\chi_2 \delta_{E_2} \theta_2), s, \psi_F) \\ &= \lambda_{E_2} \varepsilon(\sigma_1|_{W_{E_2}} \otimes (\chi_2 \delta_{E_2} \theta_2), s, \psi_{E_1}). \end{aligned}$$

Assume $l \neq p$, then π_1 can be written in the form $\chi_1 \pi_{\theta_1}$ and $\sigma_2 = \text{Ind}_{W_{E_1}}^{W_F}(\chi_1 \delta_{E_1} \theta_1)$. By the Mackey decomposition and $W_{E_1} W_{E_2} = W_F$, we have

$$(\text{Ind}_{W_{E_1}}^{W_F} \chi_1 \delta_{E_1} \theta_1)|_{W_{E_2}} = \text{Ind}_{W_{E_1 E_2}}^{W_{E_2}}(\chi_1 \delta_{E_1} \theta_1) \circ N_{E_1 E_2/E_1}.$$

Since $(\chi_1 \delta_{E_1} \theta_1) \circ N_{E_1 E_2/F}$ does not factor through $N_{E_1 E_2/E_2}$,

$$\text{Ind}_{W_{E_1 E_2}}^{W_{E_2}}((\chi_1 \delta_{E_1} \theta_1) \circ N_{E_1 E_2/E_1}) \in \mathcal{G}_{E_2}(l).$$

Therefore we have :

$$\begin{aligned} \varepsilon(\pi_1 \times \pi_2, s, \psi_F) &= \lambda_{E_2} \varepsilon(\text{Ind}_{W_{E_1 E_2}}^{W_{E_2}}(\chi_1 \delta_{E_1} \theta_1) \otimes \chi_2 \delta_{E_2} \theta_2 \circ N_{E_1 E_2/E_2}, s, \psi_{E_1}) \\ &= \lambda_{E_2} \varepsilon(\chi_2 \delta_{E_2} \theta_2 \otimes (\chi_1)_{E_1 E_2} \pi_{E_2}((\theta_1 \circ N_{E_1 E_2/E_1})), s, \psi_{E_2}). \end{aligned}$$

(The last equality follows from $\Lambda_{E_2}(\pi_{\theta_1} \circ N_{E_1 E_2/E_1}) = \text{Ind}_{W_{E_1 E_2}}^{W_{E_2}}(\theta_1 \delta_{E_1}) \circ N_{E_1 E_2/E_1}$.)

When $l = p$, it follows from Proposition 3.7 and Proposition 3.5 that

$$\begin{aligned} \Lambda_{E_2}(\sigma_1|_{W_{E_2}}) &= \mathbf{1}_{E_2}(\pi_1) \\ &= (\chi_1)_{E_1 E_2} \pi_{E_2}(\beta_1, \theta_1 \circ N_{E_1 E_2/E_1}). \end{aligned}$$

Thus we have

$$\varepsilon(\pi_1 \times \pi_2, s, \psi_F) = \lambda_{E_2} \varepsilon(\chi_2 \delta_{E_2} \theta_2 \otimes (\chi_1)_{E_1 E_2} \pi_{E_2}(\beta_1, \theta_1 \circ N_{E_1 E_2/E_1}), s, \psi_{E_2}).$$

□

By combining Theorem 2.6 and Theorem 4.1, we get the complete list of

varepsilon $(\pi_1 \times \pi_2, s, \psi_F)$ for $\pi_1 \in \mathcal{A}_F(l)$ and $\pi_2 \in \mathcal{A}_F(l')$ where l is any prime and l' is a prime $\neq l$.

Remark 4.2. By the result of [7], Theorem 4.1 may be extended to the case $\pi_1 \in \mathcal{A}_F^t(m)$ and $\pi_2 \in \mathcal{A}_F^t(n)$ where $(m, n) = 1$.

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